Optimal decision under ambiguity for diffusion processes

Sören Christensen*

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draft version!

In this paper we consider stochastic optimization problems for a risk-avers investor when the decision maker is uncertain about the parameters of the underlying process. In a first part we consider problems of optimal stopping under drift ambiguity for one-dimensional diffusion processes. Analogously to the case of ordinary optimal stopping problems for one-dimensional Brownian motions we reduce the problem to the geometric problem of finding the smallest majorant of the reward function in an two-parameter function space. In a second part we solve optimal stopping problems when the underlying process can crash down. These problems are reduced to one optimal stopping problem and one Dynkin game. An explicit example is discussed.

Keywords: optimal stopping; drift ambiguity; crash-scenario; Dynkin-games; diffusion processes

Subject Classifications: 60G40; 62L15

1 Introduction

In most articles dealing with stochastic optimization problems it is assumed that the decision maker has full knowledge of the parameter of the underlying stochastic process. In many real world situations this does not seem to be a realistic assumption. Therefore in the last years different multiple prior models were studied in the economic literature.

*Christian-Albrechts-Universität, Mathematisches Seminar, Kiel, Germany, email: christensen@math.uni-kiel.de.

Here we want to mention [10] and [13] and refer to [4] for an economic discussion and further references.

When dealing with optimization problems in this setting it is assumed that the decision maker deals with the uncertainty via a worst-case approach, that is, she optimizes her reward under the assumption that the "market" chooses the worst possible prior. This is a very natural way and we also want to follow this approach.

A very important class of stochastic optimization problems is given by optimal stopping problems. These problems arise in many different fields, e.g., in pricing American-style options, in portfolio-optimization, and in sequential statistics. Discrete time problems of optimal stopping in a multiple prior setting were first discussed in [21] and analogous results to the classical ones were proved. In this setting a generalization of the classical best choice problem was treated in detail in [8]. In continuous time the case of an underlying diffusion with uncertainty about the drift is of special interest. The general theory (including adjusted Hamilton-Jacobi-Bellman equations) was developed in [4]. Some explicit examples were given there, but no systematic way to solve them was given.

Another class of stochastic optimization problems under uncertainty was dealt with in a series of papers starting with [18]: There portfolio optimization problems were considered under the assumption that the underlying asset price process can crash down and certain (unknown) time point, see also [17] for an overview.

The aim of this article is to treat optimal stopping problems under uncertainty for underlying one-dimensional diffusion processes. These kind of problems are of special interest since they arise in many situations and allow an explicit solution. Even if the underlying process is more complicated – e.g. a multidimensional diffusion process or a process with jumps – these easy problems provide a good intuition for the original problem, which does not allow an explicit solution in most situations of interest.

The structure of this article is as follows:
In Section 2 we first review some well-known facts about the solution of ordinary optimal stopping problems for an underlying Brownian motion. This problem can be solved graphically. Then we treat the optimal stopping problem under ambiguity about the drift in a similar way: The main result is that the value function can be seen as the smallest majorant of the reward function in a two-parameter class of functions. The main tool is the use of generalized r-harmonic functions. After giving an example and characterizing the worst-case measure we generalize the results to general one-dimensional diffusion processes.

In Section 3 we introduce the optimal stopping problem under ambiguity about crashes of the underlying process in the spirit of [17]. In this situation the optimal strategy can be described by two easy strategies: One pre-crash and one post-crash strategy. These strategies can be found as solutions of a one-dimensional Dynkin-game and an ordinary optimal stopping problem, that can both be solved using standard methods. We want to point out that this model is a natural situation where Dynkin-games arise and the theory developed in the last years can be used fruitfully. As an explicit example we study the valuation of American call-options in the model with crashes. Here the post-crash

strategy is the well-known threshold-strategy in the standard Black-Scholes setting. The pre-crash strategy is of the same type, but the optimal threshold is lower.

2 Optimal stopping under drift ambiguity

2.1 Graphical solution of ordinary optimal stopping problems

Problems of optimal stopping in continuous time are well-studied and the general theory is well-developed. Nonetheless the explicit solution to such problems is often hard to find and the class of explicit examples is very limited. Most of them are generalizations of the following situation, that allows an easy geometric solution:
Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on a compact interval $[a,b]$ with absorbing boundary points a and b. We consider the problem of optimal stopping given by the value function

$$v(x) = \sup_\tau \mathbb{E}_x(g(W_\tau)\mathbb{1}_{\{\tau<\infty\}}), \qquad x \in [a,b],$$

where the reward function $g:[a,b]\to[0,\infty)$ is continuous and the supremum is taken over all stopping times w.r.t. the natural filtration for $(W_t)_{t\geq 0}$. In this case it is well-known that the value function v can be characterized as the smallest concave majorant of g, see [11]. This means that the problem of optimal stopping can be reduced to finding the smallest majorant of g in an easy class of functions. For finding the smallest concave majorant of a function g one only has to consider affine functions, i.e. for each fixed point $x \in [a,b]$ the value of the smallest concave majorant is given by

$$\inf\{h_{c,d}(x) : c,d \in \mathbb{R}, h_{c,d} \geq g\},$$

where $h_{c,d}$ is an element of the two-parameter class of affine functions of the form $h_{c,d}(y) = cy + d$. This problem can be solved geometrically, see Figure 2.1. We want to remark that this problem is indeed a semi-infinite linear programming problem:

$$\begin{aligned} \text{min!} \quad & cx + d \\ \text{s.t} \quad & cy + d \geq g(y) \quad \text{for all } y \in [a,b]. \end{aligned}$$

This gives rise to an efficient method for solving these problems, that can be generalized in an appropriate way, see [14] for an analytical solution and [5] for a numerical point of view. The example described above is important both for theory and applications of optimal stopping since by studying it one can obtain an intuition for more complex situations such as finite time horizon problems and multidimensional driving process, where numerical methods have to be used in most situations of interest.

The goal of this section is to handle optimal stopping problems with drift ambiguity for diffusion processes similarly to the ordinary case with Brownian motion discussed above, the general case is treated in Subsection 2.5. This gives rise to an easy to handle geometric method for solving optimal stopping problems under drift ambiguity explicitly.

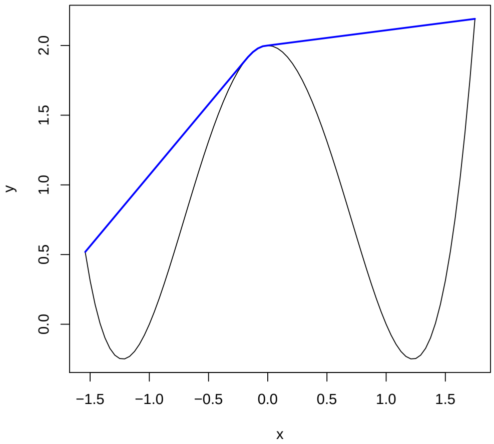

Figure 2.1: Graph of a function g (black) and its smallest concave majorant (blue)

2.2 Special Case: Brownian motion

In the following we use the notation of [4]: Let $(X_t)_{t\geq 0}$ be a Brownian motion under the measure P_0, fix $\kappa \geq 0$ and denote by $\mathcal{P}^\kappa$ the set of all probability measures, that are equivalent to P_0 with density process of the form

$$\exp\left(\int_0^t \theta_s dX_s - 1/2\int_0^t \theta_s^2 ds\right)$$

for a progressively measurable process $(\theta_t)_{t\geq 0}$ with $|\theta_t| \leq \kappa$ for all $t \geq 0$. We want to find the value function

$$v(x) = \sup_\tau \inf_{P\in\mathcal{P}^\kappa} \mathbb{E}_x^P(e^{-r\tau} g(X_\tau) 1\!\!1_{\{\tau<\infty\}})$$

for some fixed discounting rate $r > 0$ and a measurable reward function $g : \mathbb{R} \to [0, \infty)$, where $\mathbb{E}_x^P$ means taking expectation under the measure P when the process is started in x. Instead of taking affine functions as in the example discussed above we construct another class of appropriate functions based on the minimal r-harmonic functions for the Brownian motion with drift $\pm\kappa$ as follows:
Denote the roots of the equation

$$1/2z^2 - \kappa z - r = 0$$

by $\alpha_1 < 0 < \alpha_2$ and the roots of

$$1/2z^2 + \kappa z - r = 0$$

by $\beta_1 < 0 < \beta_2$. For all $c \in \mathbb{R}$ define the functions $h_c : \mathbb{R} \to \mathbb{R}$ via

$$h_c(x) = \begin{cases} \frac{\alpha_2}{\alpha_2-\alpha_1} e^{\alpha_1(x-c)} - \frac{\alpha_1}{\alpha_2-\alpha_1} e^{\alpha_2(x-c)}, & \text{if } x \leq c \\ \frac{\beta_2}{\beta_2-\beta_1} e^{\beta_1(x-c)} - \frac{\beta_1}{\beta_2-\beta_1} e^{\beta_2(x-c)}, & \text{if } x > c \end{cases}$$

and furthermore

$$h_\infty(x) = e^{\beta_2 x}, \quad h_{-\infty}(x) = e^{\alpha_1 x}.$$

For $c \in \mathbb{R}$ the function h_c is constructed by smoothly merging r-harmonic functions for the Brownian motion with drift $\pm\kappa$ at their minimum in c. Note that the set $\{h_c : c \in [-\infty,\infty]\}$ does not form a linear space for $\kappa > 0$. This is the main difference compared to the case without drift ambiguity. Therefore the standard techniques for optimal stopping are not applicable immediately.
Nonetheless this leads to the right $\mathcal{P}^\kappa$-supermartingales to work with:

Lemma 2.1. *(i) For all $a,b,x \in \mathbb{R}$ with $a \le x \le b$, $c \in [-\infty,\infty]$, $P \in \mathcal{P}^\kappa$ and $\tau = \inf\{t \ge 0 : X_t \notin [a,b]\}$ it holds that*

$$\mathbb{E}_x^P(e^{-r\tau} h_c(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) \ge h_c(x) \quad \text{and} \quad \mathbb{E}_x^{P_c}(e^{-r\tau} h_c(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) = h_c(x),$$

where the measure P_c is such that

$$dX_t = -\kappa\,\mathrm{sgn}(X_t - c)dt + dB_t^c$$

for a Brownian motion B^c under P_c.

(ii) For all $c \in [-\infty,\infty]$ and all stopping times τ it holds that

$$\mathbb{E}_x^{P_c}(e^{-r\tau} h_c(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) \le h_c(x),$$

(iii) For all $a,b,x \in \mathbb{R}$ with $a < x < b$, $P \in \mathcal{P}^\kappa$ and $\tau_a = \inf\{t \ge 0 : X_t = a\}$, $\tau_b = \inf\{t \ge 0 : X_t = b\}$ it holds that

$$\mathbb{E}_x^P(e^{-r\tau_a} h_\infty(X_{\tau_a})\mathbb{1}_{\{\tau_a<\infty\}}) \ge h_\infty(x), \qquad \mathbb{E}_x^{P_\infty}(e^{-r\tau_a} h_\infty(X_{\tau_a})\mathbb{1}_{\{\tau_a<\infty\}}) = h_\infty(x).$$

and

$$\mathbb{E}_x^P(e^{-r\tau_b} h_{-\infty}(X_{\tau_b})\mathbb{1}_{\{\tau_b<\infty\}}) \ge h_{-\infty}(x), \quad \mathbb{E}_x^{P_{-\infty}}(e^{-r\tau_b} h_{-\infty}(X_{\tau_b})\mathbb{1}_{\{\tau_b<\infty\}}) = h_{-\infty}(x).$$

Proof. (i) We can apply (the generalized version of) Itô's lemma to h_c, since the function is C^1 and C^2 up to a one-point set, see e.g. [15, p. 219]. We obtain for $P \in \mathcal{P}$ with density process θ

$$\begin{aligned} e^{-rt} h_c(X_t) =& h_c(x) + \int_0^t e^{-ru}(\kappa\,\mathrm{sgn}(X_u - c) + \theta_u) h_c'(X_u) du \\ &+ \int_0^t e^{-ru} h_c'(X_u) dW_u^P, \end{aligned}$$

where W^P is a Brownian motion under P. Noting that $(\kappa\,\mathrm{sgn}(X_u - c) + \theta_u) \ge 0$ iff $h_c'(X_u) \ge 0$, we obtain that the process $(e^{-r(t\wedge\tau)} h_c(X_{t\wedge\tau}))_{t\ge0}$ is a bounded P-submartingale. Therefore we obtain by optional sampling

$$\mathbb{E}_x^P(e^{-r\tau} h_c(X_\tau)) \ge \mathbb{E}_x^P(h_c(X_0)) = h_c(x).$$

Under P^c we see that $(e^{-r(t\wedge\tau)} h_c(X_{t\wedge\tau}))_{t\ge0}$ is even a local martingale, that is bounded. Therefore the optional sampling theorem yields equality.

(ii) By the calculation in (i) the process $(e^{-rt}h_c(X_t))_{t\geq 0}$ is a positive local P^c-martingale, i.e. also a P^c-supermartingale. The optional sampling theorem for supermartingales is applicable.

(iii) By noting that h_∞ is increasing and $h_{-\infty}$ is decreasing the same arguments as in (i) apply.

□

The following theorem shows that the geometric solution described in the introduction can indeed be generalized to the drift ambiguity case. Furthermore we give a characterization of the optimal stopping set as maximum point of explicitly given functions.

Theorem 2.2. *(i) It holds that*

$$v(x) = \inf\{\lambda h_c(x) : c \in [-\infty, \infty], \lambda \in [0, \infty], \lambda h_c \geq g\} \qquad \text{for all } x \in \mathbb{R}.$$

Furthermore the infimum in c is indeed a minimum.

(ii) A point $x \in \mathbb{R}$ is in the optimal stopping set $\{y : v(y) = g(y)\}$ if and only if there exists $c \in [-\infty, \infty]$ such that

$$x \in \operatorname{argmax} \frac{g}{h_c}.$$

Proof. Using Lemma 2.1 (ii) for each $x \in \mathbb{R}$, $c \in [-\infty, \infty]$ and each stopping time τ we obtain

$$\begin{aligned}
\inf_P \mathbb{E}_x^P(e^{-r\tau}g(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) &= \inf_P \mathbb{E}_x^P\left(e^{-r\tau}h_c(X_\tau)\frac{g}{h_c}(X_\tau)\mathbb{1}_{\{\tau<\infty\}}\right)\\
&\leq \sup\left(\frac{g}{h_c}\right)\inf_P \mathbb{E}_x^P(e^{-r\tau}h_c(X_\tau)\mathbb{1}_{\{\tau<\infty\}})\\
&\leq \sup\left(\frac{g}{h_c}\right)\mathbb{E}_x^{P_c}(e^{-r\tau}h_c(X_\tau)\mathbb{1}_{\{\tau<\infty\}})\\
&\leq \sup\left(\frac{g}{h_c}\right)h_c(x).
\end{aligned}$$

Since $\lambda h_c \geq g$ is equivalent to $\lambda \geq \sup\left(\frac{g}{h_c}\right)$ we obtain that

$$v(x) \leq \inf\{\lambda h_c(x) : c \in [-\infty, \infty], \lambda \geq 0, \lambda h_c \geq g\}.$$

For the other inequality consider the following cases:
Case 1:

$$\sup_{y\in\mathbb{R}} \frac{g(y)}{h_\infty(y)} = \sup_{y\geq x} \frac{g(y)}{h_\infty(y)}.$$

Take a sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \geq x$ such that $g(y_n)/h_\infty(y_n) \to \sup_{y\in\mathbb{R}} \frac{g(y)}{h_\infty(y)}$. Then for $\tau_n = \inf\{t \geq 0 : X_t = y_n\}$ using Lemma 2.1 (iii) we obtain

$$\begin{aligned}
v(x) &\geq \inf_P \mathbb{E}_x^P(e^{-r\tau_n} g(X_{\tau_n})\mathbb{1}_{\{\tau_n<\infty\}}) \\
&= \inf_P \mathbb{E}_x^P(e^{-r\tau_n} h_\infty(X_{\tau_n}) \frac{g}{h_\infty}(X_{\tau_n})\mathbb{1}_{\{\tau_n<\infty\}}) \\
&= \frac{g}{h_\infty}(y_n) h_\infty(x) \\
&\to \sup_{y\in\mathbb{R}} \frac{g(y)}{h_\infty(y)} h_\infty(x) \qquad \text{for } n \to \infty
\end{aligned}$$

and we obtain that $v(x) \geq \inf\{\lambda h_c(x) : c \in [-\infty, \infty], \lambda \in [0, \infty], \lambda h_c \geq g\}$.
Furthermore if x is in the stopping set, i.e. $v(x) = g(x)$, then we furthermore see that $g(x)/h_\infty(x) = \sup_{y\in\mathbb{R}} \frac{g(y)}{h_\infty(y)}$, i.e. x is a maximum point of $g(y)/h_{-\infty}$, i.e. (ii).
Case 2: The case $\sup_{y\in\mathbb{R}} g(y)/h_{-\infty}(y) = \sup_{y\leq x} g(y)/h_{-\infty}(y)$ can be handled the same way.
Case 3:

$$\sup_{y\leq x} \frac{g(y)}{h_\infty(y)} > \sup_{y\geq x} \frac{g(y)}{h_\infty(y)} \quad \text{and} \quad \sup_{y\leq x} \frac{g(y)}{h_{-\infty}(y)} < \sup_{y\geq x} \frac{g(y)}{h_{-\infty}(y)}.$$

First we show that there exists $c^* \in \mathbb{R}$ such that

$$\sup_{y\leq x} \frac{g(y)}{h_{c^*}(y)} = \sup_{y\geq x} \frac{g(y)}{h_{c^*}(y)} :$$

Since

$$\begin{aligned}
\sup_{y\leq x} \frac{g(y)}{h_c(y)} = [\min\{&e^{-\alpha_2 c} \inf_{y\leq x} (\frac{\alpha_2}{\alpha_2-\alpha_1} e^{\alpha_1 y} e^{(\alpha_2-\alpha_1)c} + \frac{\alpha_1}{\alpha_2-\alpha_1} e^{\alpha_2 y}), \\
&e^{-\beta_2 c} \inf_{y\leq x} (\frac{\beta_2}{\beta_2-\beta_1} e^{\beta_1 y} e^{(\beta_2-\beta_1)c} + \frac{\beta_1}{\beta_2-\beta_1} e^{\beta_2 y})\}]^{-1}
\end{aligned}$$

and since the functions

$$z \mapsto \inf_{y\leq x} (\frac{\alpha_2}{\alpha_2-\alpha_1} e^{\alpha_1 y} z + \frac{\alpha_1}{\alpha_2-\alpha_1} e^{\alpha_2 y})$$

and

$$z \mapsto \inf_{y\leq x} (\frac{\beta_2}{\beta_2-\beta_1} e^{\beta_1 y} z + \frac{\beta_1}{\beta_2-\beta_1} e^{\beta_2 y})$$

are continuous as concave functions we obtain that the function $c \mapsto \sup_{y\geq x} \frac{g(y)}{h_c(y)}$ is continuous . By the same argument the function $c \mapsto \sup_{y\leq x} \frac{g(y)}{h_c(y)}$ is also continuous. By the intermediate value theorem applied to the function

$$c \mapsto \sup_{y\leq x} \left(\frac{g(y)}{h_c(y)}\right) - \sup_{y\geq x} \left(\frac{g(y)}{h_c(y)}\right)$$

there exists c^* with $\sup_{y \le x} \frac{g(y)}{h_{c^*}(y)} = \sup_{y \ge x} \frac{g(y)}{h_{c^*}(y)}$ as desired.
Now take sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ with $y_n \le x \le z_n$ such that

$$\sup_{y \le x} \frac{g(y)}{h_{c^*}(y)} = \lim_{n\to\infty} \frac{g(y_n)}{h_{c^*}(y_n)} = \lim_{n\to\infty} \frac{g(z_n)}{h_{c^*}(z_n)} = \sup_{y \ge x} \frac{g(y)}{h_{c^*}(y)}.$$

Using $\tau_n = \inf\{t \ge 0 : X_t \notin [y_n, z_n]\}$ we obtain by Lemma 2.1 (i)

$$\begin{aligned} v(x) &\ge \inf_P \mathbb{E}_x^P (e^{-r\tau_n} h_{c^*}(X_{\tau_n}) \frac{g}{h_{c^*}}(X_{\tau_n}) \mathbb{1}_{\{\tau_n<\infty\}}) \\ &\ge \left(\frac{g}{h_{c^*}}(y_n) \wedge \frac{g}{h_{c^*}}(z_n)\right) \inf_P \mathbb{E}_x^P (e^{-r\tau_n} h_{c^*}(X_{\tau_n}) \mathbb{1}_{\{\tau_n<\infty\}}) \\ &= \left(\frac{g}{h_{c^*}}(y_n) \wedge \frac{g}{h_{c^*}}(z_n)\right) h_{c^*}(x) \to \sup \left(\frac{g}{h_{c^*}}\right) h_{c^*}(x). \end{aligned}$$

This yields the result (i). As above we furthermore see that if x is in the optimal stopping set, then it is a maximum point of g/h_{c^*}, i.e. (ii). □

Remark 2.3. 1. *We would like to emphasize that we do not need any continuity assumptions on g. This is remarkable, because even for the easy case described at the beginning of this section we were not able to find such a general result in the standard literature.*

2. *Some parts the proof are inspired by the ideas first described in [2]. It seems that other standard methods for dealing with optimal stopping problems for diffusions without drift ambiguity (such as Martin boundary theory as in [22], generalized concavity methods as in [9] or linear programming arguments as in [14]) are not applicable with minor modifications because of the nonlinear structure coming from drift ambiguity. A corresponding characterization of the optimal stopping points for the problem without ambiguity can be found in [6].*

2.3 Worst case prior

Theorem 2.2 leads to the value of the optimal stopping problem with drift ambiguity and also provides an easy way to find the optimal stopping time. Another important topic is to determine the worst case measure for a process started in a point x, i.e. we would like to determine the measure P such that $v(x) = \sup_\tau \mathbb{E}_x^P(e^{-r\tau} g(X_\tau) \mathbb{1}_{\{\tau<y\infty\}})$. Using the results described above this can also be found immediately:

Theorem 2.4. *Let $x \in \mathbb{R}$ and let c be a minimizer as in Theorem 2.2 (i). Then P^c is a worst case measure for the process started in x.*

Proof. This is immediate from the proof of Theorem 2.2. □

2.4 Example: American straddle in the Bachelier market

Because it is easy and instructive we consider the example discussed in [4] in the light of our method:
We consider a variant of the American straddle option in a Bachelier market model as follows: As a driving process we consider a standard Brownian motion under P^0 with reward function $g(x) = |x|$. Our aim is to find the value in 0 of the optimal stopping problem

$$\sup_{\tau} \min_{P \in \mathcal{P}^\kappa} \mathbb{E}^P(e^{-r\tau}|X_\tau|\mathbb{1}_{\{\tau<\infty\}}).$$

Using Theorem 2.2 we have to find the majorant of $|\cdot|$ in the set

$$\{\lambda h_c : c \in [-\infty,\infty], \lambda \in [0,\infty], \lambda h_c \geq g\}$$

with minimal value in 0. One immediately sees that if $\lambda h_c(\cdot) \geq |\cdot|$, then $\lambda h_0(\cdot) \geq |\cdot|$ and furthermore $\lambda h_0(0) \leq \lambda h_c(0)$. Therefore we only have to consider majorants of $|\cdot|$ in the set

$$\{\lambda h_0 : \lambda \in [0,\infty], \lambda h_0 \geq g\}.$$

This one-dimensional problem can be solved immediately. For $\lambda = \max(|\cdot|/h_0(\cdot))$ one obtains $v(0) = \lambda h_0(0)$.
In fact if $-b, b$ denote the maximum points of $|\cdot|/h_0(\cdot)$ we obtain that $v(x) = \lambda h_0(x)$ for $x \in [-b,b]$. Moreover for $x \notin [-b,b]$ one immediately sees that there exists $c \in \mathbb{R}$ such that x is a maximum point of $|\cdot|/h_c(\cdot)$ and we obtain

$$v(x) = \begin{cases} \lambda h_0(x), & \text{if } \ x \in [-b,b] \\ |x|, & \text{else.} \end{cases}$$

2.5 General diffusion processes

In the case of general one-dimensional diffusion processes the only problem is to choose appropriate functions h_c carefully. After these functions are constructed the same arguments as in the previous sections work.
We consider that the process $(X_t)_{t\geq 0}$ is a one-dimensional diffusion process on some interval I with boundary points $a < b, a, b \in [-\infty,\infty]$, that is characterized by its generator

$$A = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

for some continuous functions $\sigma > 0, \mu$. Again denote by $\mathcal{P}^\kappa$ the set of all probability measures, that are equivalent to P_0 with density process of the form

$$\exp\left(\int_0^t \theta_s dX_s - 1/2\int_0^t \theta_s^2 ds\right)$$

for a progressively measurable process $(\theta_t)_{t\geq 0}$ with $\theta_t \leq \kappa$ for all $t \geq 0$. We denote the fundamental solutions of the equation

$$\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}\psi + (\mu(x)+\kappa)\frac{d}{dx}\psi = r\psi$$

by ψ_+^κ resp. ψ_-^κ for the increasing resp. decreasing positive solution, cf. [3, II.10] for a discussion and further references. Analogously denote the fundamental solutions of

$$\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}\psi+(\mu(x)-\kappa)\frac{d}{dx}\psi=r\psi$$

by $\psi_+^{-\kappa}$ resp. $\psi_-^{-\kappa}$. Note that for each positive solution ψ of one of the above ODEs it holds that

$$\frac{d^2}{dx^2}\psi(x)=\frac{(\mu(x)\pm\kappa)\frac{d}{dx}\psi(x)+r\psi(x)}{\frac{1}{2}\sigma^2(x)},$$

hence all extremal points are minima, so that ψ has at most one minimum. Therefore for each $s\in(0,1)$ the function $\psi=s\psi_+^{\pm\kappa}+(1-s)\psi_-^{\pm\kappa}$ has a unique minimum point and furthermore each $c\in E$ arises as such a minimum point. Therefore for each $c\in(a,b)$ we can find constants $\gamma_1,...,\gamma_4$ such that the function

$$h_c:E\to\mathbb{R},x\mapsto\begin{cases}\gamma_1\psi_+^\kappa(x)+\gamma_2\psi_-^\kappa(x), & \text{if } x\le c\\ \gamma_3\psi_+^{-\kappa}(x)+\gamma_4\psi_-^{-\kappa}(x), & \text{if } x>c\end{cases}$$

is C^1 with a unique minimum point in c. Furthermore write $h_a=\psi_-^\kappa$ and $h_b=\psi_+^{-\kappa}$. Now all the arguments given in the previous section apply and we again obtain the following result (compare Theorem 2.2):

Theorem 2.5. *(i) It holds that*

$$v(x)=\inf\{\lambda h_c(x):c\in[a,b],\lambda\in[0,\infty],\lambda h_c\ge g\}\qquad \textit{for all } x\in I.$$

(ii) A point $x\in E$ is in the optimal stopping set $\{y:v(y)=g(y)\}$ if and only if there exists $c\in[a,b]$ such that

$$x\in\operatorname{argmax}\frac{g}{h_c}.$$

3 Optimal decision for models with crashes

Now denote by Y a one-dimensional regular diffusion process on an interval I. Denote by $\mathcal{F}$ the natural filtration generated by Y. In this section we assume that all parameters of this process are known. This process represents the asset price process of the underlying asset if no crash occurs; therefore for economical plausibility it is reasonable to assume $I=(0,\infty)$.

Now we modify the process such that at a certain random time point σ a crash of hight $\zeta\in[0,c]$ occurs, where $c\in(0,1)$ is a given constant, that described an upper bound for the hight of the crash.

To be more precise: For a given stopping time σ and an $\mathcal{F}_\sigma$-measurable and $[c,1]$-valued random variable ζ we consider the modified process $X^{\sigma,\zeta}$ given by

$$X_t^{\sigma,\zeta}=\begin{cases}Y_t & t\le\sigma\\ \zeta Y_t & t>\sigma.\end{cases}$$

Now we consider the optimal stopping problem connected to the pricing of perpetual American options in this market, i.e. let $g:(0,\infty)\to[0,\infty)$ be a continuous reward function. We furthermore assume g to be non-decreasing, so that a crash always leads to a lower payoff of the option. We assume that $r>0$ in the following and that g is non-increasing. This is a natural assumptions for many situations of interest.
We furthermore assume that the holder of the option does know that the process will crash once in the future, but she does not know more about (σ,ζ). But we assume the crash to be observable for the investor, so she will specify her action by a pre-crash stopping-time $\underline{\tau}$ and a post-crash stopping time $\overline{\tau}$, i.e. given σ she takes the strategy

$$\tau=\tau_\sigma=\begin{cases}\underline{\tau}, & \underline{\tau}\leq\sigma\\ \sigma+\overline{\tau}\circ\theta_\sigma, & \text{else,}\end{cases} \tag{1}$$

where $\theta.$ denotes the time-shift operator. As before we assume that the holder of the option is risk-avers and hence wants to maximize her expected reward under the worst scenario, i.e. she tries to solve the problem

$$v(x)=\sup_{\underline{\tau},\overline{\tau}}\inf_{\sigma,\zeta}\mathbb{E}_x(e^{-r\tau}g(X_\tau^{\sigma,\zeta})), \tag{2}$$

where $\tau=\tau_\sigma$ is given as in (1).

Remark 3.1. *Obviously by the monotonicity of the reward function we always have*

$$v(x)=\sup_{\underline{\tau},\overline{\tau}}\inf_{\sigma}\mathbb{E}_x(e^{-r\tau}g(X_\tau^{\sigma}))$$

where $X^\sigma:=X^{\sigma,c}$.

We obtain the following reduction of the optimal stopping problem under ambiguity about the crashes: It shows that the problem can be reduced into one optimal stopping problem and one Dynkin game for the diffusion process Y (without crashes).

Theorem 3.2. *(i) It holds that*

$$v(x)=\sup_{\underline{\tau}}\inf_{\sigma}\mathbb{E}_x(e^{-r\underline{\tau}}g(Y_{\underline{\tau}})\mathbb{1}_{\{\underline{\tau}\leq\sigma\}}+e^{-r\sigma}\hat{g}(Y_\sigma)\mathbb{1}_{\{\underline{\tau}>\sigma\}}), \tag{3}$$

where $\hat{g}$ is the value function for the optimal stopping problem for cY with reward g, i.e.

$$\hat{g}(y)=\sup_{\overline{\tau}}\mathbb{E}_y(e^{-r\overline{\tau}}g(cY_{\overline{\tau}}))\text{ for all } y\in(0,\infty). \tag{4}$$

(ii) Furthermore if $\overline{\tau}$ is optimal for (4) and $\underline{\tau},\sigma$ is a Nash-equilibrium for (3), then $(\underline{\tau},\overline{\tau})$, (σ,c) is a Nash-equilibrium for (2).

Proof. (i) Firstly fix $\underline{\tau}$, $\overline{\tau}$. Then for all σ by conditioning on $\mathcal{F}_\sigma$ we obtain

$$\begin{aligned}\mathbb{E}_x(e^{-r\tau}g(X_\tau^\sigma))&=\mathbb{E}_x(e^{-r\underline{\tau}}g(Y_{\underline{\tau}})\mathbb{1}_{\{\underline{\tau}\leq\sigma\}}+e^{-r(\sigma+\overline{\tau}\circ\theta_\sigma)}g(cY_{\sigma+\overline{\tau}\circ\theta_\sigma})\mathbb{1}_{\{\underline{\tau}>\sigma\}})\\&=\mathbb{E}_x(e^{-r\underline{\tau}}g(Y_{\underline{\tau}})\mathbb{1}_{\{\underline{\tau}\leq\sigma\}}+e^{-r\sigma}\mathbb{E}_x(e^{-r(\overline{\tau}\circ\theta_\sigma)}g(cY_{\sigma+\overline{\tau}\circ\theta_\sigma})|\mathcal{F}_\sigma)\mathbb{1}_{\{\underline{\tau}>\sigma\}}).\end{aligned}$$

By the strong Markov property we furthermore obtain

$$\mathbb{E}_x(e^{-r(\overline{\tau}\circ\theta_\sigma)}g(cY_{\sigma+\overline{\tau}\circ\theta_\sigma})|\mathcal{F}_\sigma) = \mathbb{E}_{Y_\sigma}(e^{-r\overline{\tau}}g(cY_{\overline{\tau}})) \leq \hat{g}(Y_\sigma).$$

Therefore

$$\mathbb{E}_x(e^{-r\tau}g(X_\tau^\sigma)) \leq \mathbb{E}_x(e^{-r\underline{\tau}}g(Y_{\underline{\tau}})\mathbb{1}_{\{\underline{\tau}\leq\sigma\}} + e^{-r\sigma}\hat{g}(Y_\sigma)\mathbb{1}_{\{\underline{\tau}>\sigma\}}),$$

showing that

$$v(x) \leq \sup_{\underline{\tau}}\inf_{\sigma}\mathbb{E}_x(e^{-r\underline{\tau}}g(Y_{\underline{\tau}})\mathbb{1}_{\{\underline{\tau}\leq\sigma\}} + e^{-r\sigma}\hat{g}(Y_\sigma)\mathbb{1}_{\{\underline{\tau}>\sigma\}}).$$

Now take a sequence of $1/n$-optimal stopping times $(\overline{\tau}_n)_{n\in\mathbb{N}}$ for the problem (4), i.e.

$$\hat{g}(y) \leq \mathbb{E}_y(e^{-r\overset{*}{\tau}_n}g(cY_{\overline{\tau}_n})) + \frac{1}{n} \text{ for all } n \in \mathbb{N} \text{ and all } y.$$

Then

$$\mathbb{E}_{Y_\sigma}(e^{-r\overline{\tau}_n}g(cY_{\overline{\tau}_n})) \leq \hat{g}(Y_\sigma) + \frac{1}{n},$$

and hence considering the post-crash strategy $\overline{\tau}_n$ and arbitrary $\underline{\tau}, \sigma$ we see that

$$v(x) + \frac{1}{n} \geq \sup_{\underline{\tau}}\inf_{\sigma}\mathbb{E}_x(e^{-r\underline{\tau}}g(Y_{\underline{\tau}})\mathbb{1}_{\{\underline{\tau}\leq\sigma\}} + e^{-r\sigma}\hat{g}(Y_\sigma)\mathbb{1}_{\{\underline{\tau}>\sigma\}}),$$

proving equality.

(ii) is obvious by the proof of (i).

□

Remark 3.3. *Note that the arguments used so far have nothing to do with diffusion processes, but can be applied in the same way for general one-dimensional strong Markov processes, like one-dimensional Hunt processes. Nonetheless we decided to consider this more special setup because of its special importance and since the theory for explicitly solving optimal stopping problems and Dynkin games is so well established. Nonetheless we would like to mention [7] for the explicit solution of optimal stopping problems in the more general setting.*

The previous reduction theorem solves the optimal stopping problem (2) since both problems (3) and (4) are well-studied for diffusion processes, see e.g. the references given above for optimal stopping problems and [12], [1] and [20] for Dynkin games. It is interesting to see that the optimal stopping problem under crash-scenarios naturally leads to Dynkin games as studied extensively in the last years. The financial applications studied so far were based on Israeli options, that are (at least at first glance) of a different nature, see [16].

3.1 Example: Perpetual Call-option with crashes

As an example we consider a geometric Brownian motion given by the dynamics

$$dX_t = X_t(\mu dt + \sigma dW_t), \quad t \geq 0$$

and we take $g:(0,\infty) \to [0,\infty)$ given by $g(x) = (x-K)^+$, where $K > 0$ is a constant. To exclude trivial cases we assume that $\mu < r$. Then a closed-form solution of the optimal stopping problem

$$\hat{g}(y) = \sup_\tau \mathbb{E}_y(e^{-r\tau}(cY_\tau - K)^+) = \sup_\tau \mathbb{E}_{cy}(e^{-r\tau}(Y_\tau - K)^+)$$

is well known (see e.g. [19] for the more general case of exponential Lévy processes) and is given by

$$\hat{g}(y) = \begin{cases} cy - K, & cy \geq x^* \\ d(cy)^\gamma, & cy < x^*, \end{cases}$$

where γ is the positive solution to

$$\frac{\sigma^2}{2} z^2 + \left(\mu - \frac{\sigma^2}{2}\right) z + r = 0$$

and x^* and d are chosen such that the smooth fit principle holds at x^*. Furthermore the optimal stopping time is given by $\overline{\tau} := \inf\{t \geq 0 : X_t \geq x^*\}$. By Theorem 3.2 we are faced with the Dynkin game

$$v(x) = \sup_{\underline{\tau}} \inf_\sigma \mathbb{E}_x(e^{-r\underline{\tau}}(Y_{\underline{\tau}} - K)^+ \mathbb{1}_{\{\underline{\tau} \leq \sigma\}} + e^{-r\sigma} \hat{g}(Y_\sigma) \mathbb{1}_{\{\underline{\tau} > \sigma\}}), \tag{5}$$

To solve this problem first note that there exists $x' \in (K, x^*/c)$ such that $g(x) \leq \hat{g}(x)$ for $x \in (0, x']$ and $g(x) \geq \hat{g}(x)$ for $x \in [x', \infty)$; indeed x' is the unique positive solution to

$$d(cx)^\gamma = x - K,$$

see Figure 3.1.

We could use the general theory to solve the optimal stopping game (5), but we can also solve it elementary here:
First let $x > x'$. Then for all stopping times σ^* with $\sigma^* = 0$ under P_x and each stopping time $\underline{\tau}$ we obtain

$$\mathbb{E}_x(e^{-r\underline{\tau}}(Y_{\underline{\tau}} - K)^+ \mathbb{1}_{\{\underline{\tau} \leq \sigma^*\}} + e^{-r\sigma^*} \hat{g}(Y_{\sigma^*}) \mathbb{1}_{\{\underline{\tau} > \sigma^*\}}) = \mathbb{E}_x(g(x) \mathbb{1}_{\{\underline{\tau} = 0\}} + \hat{g}(x) \mathbb{1}_{\{\underline{\tau} > 0\}}) \leq g(x)$$

with equality if $\tau = 0$ P_x-a.s. On the other hand for $\tau^* = 0$ the payoff is $g(x)$, independent of σ.

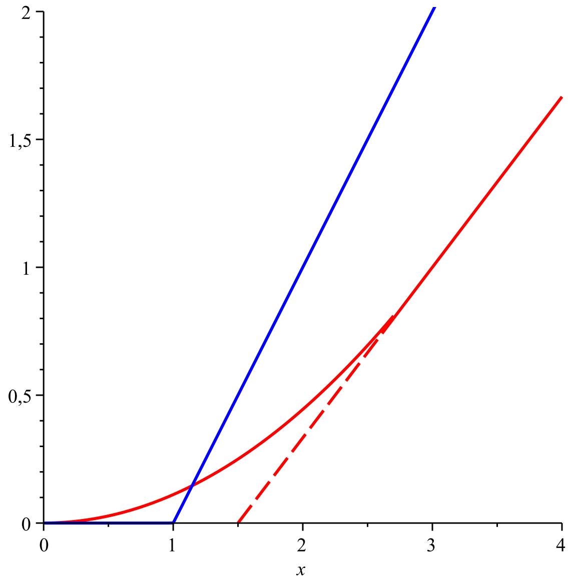

Figure 3.1: Graphs of g (blue) and $\hat{g}$ (red).

For $x \le x'$ by by taking $\underline{\tau} = \inf\{t \ge 0 : X_t \ge x'\}$ we have for each stopping time σ by definition of x':

$$\begin{aligned}
&\mathbb{E}_x(e^{-r\underline{\tau}}(Y_{\underline{\tau}} - K)^+ \mathbb{1}_{\{\underline{\tau}\le\sigma\}} + e^{-r\sigma}\hat{g}(Y_\sigma)\mathbb{1}_{\{\underline{\tau}>\sigma\}}) \\
=&\mathbb{E}_x(e^{-r\underline{\tau}}(x' - K)\mathbb{1}_{\{\underline{\tau}\le\sigma\}} + e^{-r\sigma}d(cY_\sigma)^\gamma\mathbb{1}_{\{\underline{\tau}>\sigma\}}) \\
=&\mathbb{E}_x(e^{-r\underline{\tau}}d(cx')^\gamma\mathbb{1}_{\{\underline{\tau}\le\sigma\}} + e^{-r\sigma}d(cY_\sigma)^\gamma\mathbb{1}_{\{\underline{\tau}>\sigma\}}) \\
=&\mathbb{E}_x(e^{-r(\sigma\wedge\underline{\tau})}d(cY_{\sigma\wedge\underline{\tau}})^\gamma) \\
=&dc^\gamma\mathbb{E}_x(e^{-r(\sigma\wedge\underline{\tau})}(Y_{\sigma\wedge\underline{\tau}})^\gamma) \\
=&dc^\gamma x^\gamma = \hat{g}(x),
\end{aligned}$$

where the last step holds by the fundamental properties of the minimal r-harmonic functions, see e.g. [3, II.9]. By taking any stopping time $\sigma^* \le \inf\{t \ge 0 : y_t \ge x'\}$ and any stopping time τ the same calculation holds.
Putting pieces together we obtain that $\underline{\tau}, \sigma^*$ is a Nash-equilibrium of the Dynkin-game (5) for any stopping time $\sigma^* \le \underline{\tau}$.
By applying Theorem 3.2 we get

Proposition 3.4. *The value function v is given by*

$$v(x) = \begin{cases} x - K, & x \ge x' \\ d(cx)^\gamma, & x < x', \end{cases}$$

and for

$$\underline{\tau} = \inf\{t \ge 0 : Y_t \ge x'\} \qquad \text{'pre-crash strategy'}$$

and

$$\overline{\tau} = \inf\{t \geq 0 : X_t \geq x^*\} \qquad \text{'post-crash strategy'}$$

and any stopping time $\sigma^* \leq \tau$ *it holds that* $(\underline{\tau}, \overline{\tau})$, (σ^*, c) *is a Nash-equilibrium of the problem.*

The solution to this example is very natural: If the investor expects a crash in the market, then she exercises the option as soon as the asset price reaches the level x' (pre-crash strategy). After the crash, i.e. if the investor does not expect to have one more crash, then she takes the ordinary stopping time, i.e. she stops if the process reaches level $x^* > x'$ (post-crash strategy).

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